ON THE STABILITY OF SOME PARTICULAR CASES OF MOTION OF A SYMMETRICAL GYROSCOPE CONTAINING LIQUID MASS

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In this note particular cases of the motion of a gyroscope containing an ellipsoidal cavity filled with an ideal, incompressible liquid are considered and necessary and sufficient conditions are established for the stability of such motions.

1. The equations of motion of a symmetrical gyroscope with an ellipsoidal cavity filled with an ideal incompressible liquid, whose vorticity is assumed to be constant for all its points, were obtained in different forms, under different assumptions concerning the acting forces and the cavity location, by Greenhill [1], Hough [2], Poincare [3], Zhukovskii [4] and others.* If no forces are present except gravity, then after the introduction of the Zhukovskii function Ψ the motion of the system can be described by two ordinary vector differential equations

$$\frac{d\mathbf{H}}{dt} = (\mathbf{H}\cdot\nabla)\,\nabla\Psi\cdot(\Omega - \mathbf{H}), \qquad \frac{d\mathbf{W}}{dt} = \mathbf{L}, \qquad \mathbf{H}(t) = {}^1\!/_2 \operatorname{rot} \mathbf{v} \tag{1.1}$$

Here $\Omega\ (t)$ is the instantaneous angular velocity of the gyroscope and $\mathbf{L}$ is the angular momentum of the moment of the gravity forces, the system has the form:

$$\mathbf{W} = J_0\cdot\Omega + J\cdot\mathbf{H} + J_1\cdot(\Omega - \mathbf{H}) = (J_0 + J_1)\cdot\Omega + (J - J_1)\cdot\mathbf{H} \tag{1.2}$$

where J_0 is the inertia tensor of the shell, J the inertia tensor of the liquid mass in the cavity and J_1 the inertia tensor of the "equivalent" rigid body corresponding to the liquid in the cavity [4].

* After submitting this note to the editors, I learned about the article by V.V. Rumiantsev, "On the stability of the rotation of a rigid body containing an ellipsoidal cavity filled with liquid", Trudy of the Mechanics Institute of the Academy of Sciences USSR, No.2, 1956, also devoted to the problem investigated.

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If we consider the "astatic" (semi-fixed) system of coordinates $O\xi\eta\zeta$ (Fig. 1) of the gyroscope, which does not take part in the rotation of the gyroscope about the common axis of symmetry of the shell and of the cavity $O\zeta$, then the tensors $J_0 + J_1$, $J - J_1$ have only the constant diagonal components A_1, C_1 and A_2, C_2, respectively.

Denoting the projections of $\mathbf{H}$ and Ω along this set of axes by p_1, q_1, r_1, p, q, r, we obtain the following equations of motion

$$\dot{p}_1 = r_1 [\varepsilon q_1 - (1+\varepsilon) q]$$
$$\dot{q}_1 = -r_1 [\varepsilon p_1 - (1+\varepsilon) p] \quad \left(\varepsilon = \frac{a^2 - c^2}{a^2 + c^2}\right) \tag{1.3}$$
$$\dot{r}_1 = (1-\varepsilon)(p_1 q - q_1 p)$$

$$A_1\dot{p} + A_2\dot{p}_1 + q(C_1 r + C_2 r_1) = L_1$$
$$A_1\dot{q} + A_2\dot{q}_1 - p(C_1 r + C_2 r_1) = L_2 \tag{1.4}$$
$$C_1\dot{r} + C_2\dot{r}_1 - A_2(p_1 q - q_1 p) = 0$$

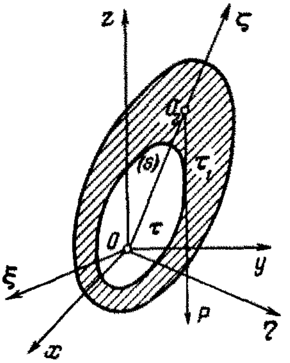

Fig. 1.

(ϵ is the eccentricity of the ellipsoid, $a = b$, c are the semi-axes of the ellipsoid).

Since we have for the ellipsoid of revolution

$$A_2 = (1-\varepsilon) C_2 = \frac{4Ma^2c^2}{5(a^2+c^2)}$$

where M is the mass of liquid in the cavity, it follows from the third equation of (1.3) and (1.4)

$$C_1\dot{r} = 0, \qquad r = \text{const} = r_0$$

Denoting the direction cosines of the vertical (line of action of the weight vector) by γ_1, γ_2, γ_3 in the selected coordinate system, the weight of the system by P, and the distance from the center of gravity of the system to the fixed point by h, we can then write

$$\mathbf{L} = Ph\boldsymbol{\xi}^\circ \times \mathbf{g}^\circ \qquad L_1 = Ph\gamma_2, \qquad L_2 = -Ph\gamma_1$$

The kinematic equations must be added for completeness

$$\dot{\gamma}_1 = -q\gamma_3, \qquad \dot{\gamma}_2 = p\gamma_3, \qquad \dot{\gamma}_3 = q\gamma_1 - p\gamma_2 \tag{1.5}$$

Equations (1.3) to (1.5) define completely the motion of the gyroscope for a given initial position and initial angluar velocities of the system.

2. If the cavity is spherical, then $\epsilon = 0$, $A_2 = C_2 = 2/5 Ma^2$, therefore after elimination of p_1, q_1, equations (1.4) become

$$A_1\dot{p} + C_1 r_0 q = Ph\gamma_2, \qquad A_1\dot{q} - C_1 r_0 p = -Ph\gamma_1 \tag{2.1}$$

i.e., they become independent of p_1, q_1, r_1. Equations of motion of the rigid gyroscope, in the selected coordinate system, have then the form

$$A\dot{p} + Cr_0 q = Ph\gamma_2, \qquad A\dot{q} - Cr_0 p = -Ph\gamma_1, \qquad A = A_1 + A_2, \qquad C = C_1 + C_2 \qquad (2.2)$$

Equations (1.5) remain the same. As was established by Chetaev [6], a necessary and sufficient condition for the stability of motion of a rigid gyroscope rotating about the vertical position of equilibrium is given by the inequality

$$r_0^2 > \frac{4APh}{C^2} = r_{01}^2$$

Repeating the above reasoning and calculations in its entirety, we can obtain the necessary and sufficient condition for the stability of motion of a symmetrical gyroscope with a spherical cavity filled with an ideal, incompressible liquid, rotating about its vertical position of equilibrium ($p = q = 0$, $r = r_0$, $\gamma_1 = \gamma_2 = 0$, $\gamma_3 = 1$). The stability condition is expressed by the inequality

$$r_0^2 > \frac{4A_1Ph}{C_1^2} = r_{02}^2$$

If the moments of inertia of the system are replaced by the moments of inertia of the "transformed rigid body" [4], the system will behave as a rigid gyroscope which does not contain a liquid mass.

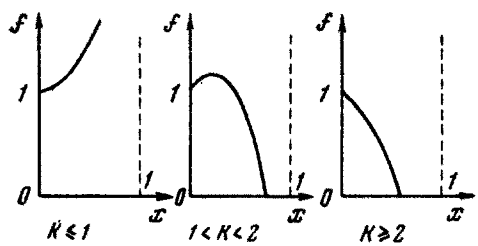

Fig. 2.

It is of interest to estimate the influence of the liquid by investigating the ratio of the critical angular velocities

$$f(k, x) = \left(\frac{r_{02}}{r_{01}}\right)^2 = \frac{A_1C^2}{AC_1^2} = \frac{1 - kx}{(1-x)^2} \qquad \left(k = \frac{C}{A}, \quad x = \frac{C_2}{C}\right) \qquad (2.3)$$

Graphs expressing such a relationship are given in Fig. 2. It follows from the diagrams that, for example, in the simplest case in which the density of the shell and of the liquid are the same, the critical angular velocity r_{02} increases with an increase in the cavity size if $C \leqslant A$ ($k \leqslant 1$) (i.e. A, C is constant, therefore k remains constant). If $A < C < 2A$ ($1 < k < 2$) with respect to r_{01}, the critical angular velocity increases at first and then decreases. The condition $k \geqslant 2$ does not exist because $2A > C$ always.

3. In the case of a very thin shell, its moment of inertia can be neglected entirely, and if the fixed point is located at the center of the cavity, it is again possible to apply the Chetaev method [5-7]. In this case

$$A_1 = \tfrac{1}{5} M (a^2 + c^2) \varepsilon^2, \qquad C_1 = 0, \qquad C_2 = \tfrac{2}{5} M a^2, \qquad A_2 = (1 - \varepsilon) C_2$$

and the equations (1.4) reduce to

$$p = -\frac{1+\varepsilon}{\varepsilon} r_1 [q_1 (1-\varepsilon) + \varepsilon q], \qquad q = \frac{1+\varepsilon}{\varepsilon} r_1 [p_1 (1-\varepsilon) + \varepsilon p] \tag{3.1}$$

Equations (1.3 do not change and equations (1.5), after determining p and q from (3.1) and (1.3), permit us subsequently to find γ_1, γ_2, γ_3.

Equations (3.1) and (1.3) give the following first integrals

$$p^2 + q^2 - \frac{1+\varepsilon}{\varepsilon} r_1^2 = \text{const}$$

$$p_1^2 + q_1^2 + \frac{1+\varepsilon}{1-\varepsilon} r_1^2 = \text{const} \tag{3.2}$$

$$2pp_1 + 2qq_1 + \frac{1+2\varepsilon}{1-\varepsilon} r_1^2 = \text{const}$$

or by elimination of r_1, we obtain

$$p^2 + q^2 + \frac{1-\varepsilon}{\varepsilon}(p_1^2 + q_1^2) = \text{const}, \quad 2pp_1 + 2qq_1 - \frac{1+2\varepsilon}{1+\varepsilon}(p_1^2 + q_1^2) \equiv V_2 = \text{const} \tag{3.3}$$

Taking a linear combination of the above integrals

$$V = V_1 + \lambda V_2 = p^2 + 2\lambda pp_1 + p_1^2 \left(\frac{1-\varepsilon}{\varepsilon} - \lambda \frac{1+2\varepsilon}{1+\varepsilon}\right) + q^2 + 2\lambda qq_1 + $$
$$+ q_1^2 \left(\frac{1-\varepsilon}{\varepsilon} - \lambda \frac{1+2\varepsilon}{1+\varepsilon}\right) \tag{3.4}$$

we see that such a quadratic form, being constant by virtue of the equations of motion, will be positive-definite if, and only if

$$\begin{vmatrix} 1 & \lambda \\ \lambda & \frac{1-\varepsilon}{\varepsilon} + \frac{1+2\varepsilon}{1+\varepsilon}\lambda \end{vmatrix} = -\lambda^2 - \lambda \frac{1+2\varepsilon}{1+\varepsilon} + \frac{1-\varepsilon}{\varepsilon} > 0$$

In order to select λ so that the inequality would be satisfied, the following relation must hold

$$D = \left(\frac{1+2\varepsilon}{1+\varepsilon}\right)^2 + 4\frac{1-\varepsilon}{\varepsilon} = \frac{4+5\varepsilon}{\varepsilon(1+\varepsilon)^2} > 0$$

$$\text{i.e. either } \varepsilon > 0 \quad (a^2 > c^2), \qquad \text{or} \quad \varepsilon < -\frac{4}{5} \quad (c^2 > 9a^2) \tag{3.5}$$

In the light of the results obtained by N.G. Chetaev [5, 6], this appears to be a necessary and sufficient condition for the stability, with respect to the variables p, q, p_1, q_1, of the motion of the gyroscope under consideration about the vertical position of equilibrium. Such conditions were obtained before by Hough [2] as conditions for the stability of the linearized equations, i.e. the equations of the first approximation.

4. If the gyroscope oscillates about the vertical position of equilibrium, then the linearization of equations (1.3) and (1.4) leads to the following. Considering p_1, q_1, p, q to be small and neglecting their products we obtain

$$\dot{r}_1 = 0, \qquad \text{or} \qquad r_1 = \text{const} = r_0 \tag{4.1}$$

Moreover, denoting the position of the gyroscope axes by the angles α and β between the vertical and the planes $\xi O \zeta$ and $\eta O \zeta$, respectively, we can also disregard their squares and products because the system $O\xi\eta\zeta$ is astatic and does not participate in the gyroscope rotation about its own axis. By assumption, they remain small during the entire duration of the motion. As a result, we have

$$p = \dot{\beta}, \qquad q = -\dot{\alpha}, \qquad \gamma_1 \approx \alpha, \qquad \gamma_2 \approx \beta, \qquad \gamma_3 \approx 1 \tag{4.2}$$

Let us substitute these values into (1.3) and (1.4)

$$\begin{aligned} &\dot{p}_1 = r_0[\varepsilon q_1 + (1+\varepsilon)\dot{\alpha}], \qquad \ddot{\beta} + \varepsilon(1-\varepsilon) k_2 r_0 q_1 - r_0 (k_1 + \varepsilon^2 k_2)\dot{\alpha} = k_3 \beta \\ &\dot{q}_1 = -r_0[\varepsilon p_1 - (1+\varepsilon)\dot{\beta}], \qquad \ddot{\alpha} + \varepsilon(1-\varepsilon) k_2 r_0 p_1 + r_0 (k_1 + \varepsilon^2 k_2)\dot{\beta} = k_3 \alpha \end{aligned} \tag{4.3}$$

where

$$k_1 = \frac{C_1}{A_1}, \quad k_2 = \frac{C_2}{A_1}, \quad k_3 = \frac{Ph}{A_1}$$

We now introduce the complex variable $Z = \alpha \pm i\beta = Ne^{\pm\lambda t}$. Eliminating p_1, q_1, we now obtain the following characteristic equation for (4.3):

$$\lambda^3 + r_0(\varepsilon - k_1 - \varepsilon^2 k_2)\lambda^2 + [k_3 - r_0^2 \varepsilon (k_1 + k_2)]\lambda + \varepsilon r_0 k_3 = 0 \tag{4.4}$$

Since an unstable first approximation corresponds to complex roots of (4.4) and a periodically stable first approximation to real roots, then on the basis of the well-known Liapunov theorem [8], the presence of complex roots in (4.4) is a sufficient condition for instability of gyroscope motion as a whole. In the presence of real roots, we are dealing with a critical case - three pairs of purely imaginary roots. The sufficient condition for stability, therefore, well be expressed by the inequality

$$f(\mu) = \mu^3 + a_1\mu^2 + a_2\mu + a_3 > 0 \tag{4.5}$$

which represents the transformed condition of the occurrence of complex roots in equation (4.4). Here the notation was introduced

$$\mu = \frac{k_3}{r_0^2}, \qquad \pi_1 = k_1 + \varepsilon^2 k_2 - \varepsilon, \qquad \pi_2 = k_1 + k_2$$

$$a_1 = \frac{1}{4}[27\varepsilon^2 + 6\varepsilon(3\pi_1 - 2\pi_2) - \pi_1^2],$$

$$a_2 = \frac{\varepsilon}{2}[3\varepsilon\pi_2(2\pi_2 - 3\pi_1) + \pi_1^2(\pi_2 - 2\pi_1)], \qquad a_3 = -\frac{\varepsilon^2\pi_2^2}{4}(\pi_1^2 + 4\varepsilon\pi_2)$$

It is of interest to point out the fact that the characteristic equation is of the third, and not of the second degree, so that the stable motion of the gyroscope axis, besides precession and nutation, will also perform a "nutation of the second order". This means that the axis rotates with constant angular velocity λ_3 around the center O_3, which in turn rotates with angular velocity λ_2 around the center O_2 and the latter rotates with angular velocity λ_1 around the projection of the gyroscope axis in the position O_1 of the so-called "sleeping top". [λ_1, λ_2, λ_3 are roots of (4.4).]

In the particular case of the spherical cavity $\epsilon = 0$, $a_2 = a_3 = 0$, $a_1 = -\frac{1}{4} k_1^2$, equation (4.5) can be written in the form

$$f(\mu) = \mu^2\left(\mu - \frac{k_1^2}{4}\right) > 0$$

i.e. the motion is unstable when

$$\mu > \frac{k_1^2}{4}, \quad \text{- or} \qquad r_0^2 < \frac{4k_3}{k_1^2} = \frac{4PhA_1}{c_1^2} = r_{02}^2$$

As was shown before, this sufficient condition for instability is also a necessary one.

In the case of the "inertialess" shell ($k_1 = 0$, $k_2 = (1+\epsilon)/\epsilon^2$, $k_3 = 0$), equation (4.4) changes into the new equation

$$\lambda\left(\lambda^2 - \lambda r_0 - r_0^2 \frac{1+\varepsilon}{\varepsilon}\right) = 0, \qquad \lambda_1 = 0,\ \lambda_2 = \frac{r_0}{2}\left(1 \pm \sqrt{\frac{4+5\varepsilon}{\varepsilon}}\right)$$

Thus we obtain again the same instability condition as in Section 3:

$$-\frac{4}{5} < \varepsilon < 0 \quad (a < c < 3a)$$

5. Let us look now into the problem of the nearly spherical cavity. To be definite, we take a sphere equal in volume to the investigated ellipsoid. Considering the coefficient ϵ to be small, we expand the coefficients of the inequality (4.5) in a power series in ϵ and then seek its roots also as a power series in ϵ; we finally obtain the following

$$f(\mu) = (\mu - \mu_1)(\mu - \mu_2)(\mu - \mu_3) > 0 \tag{5.1}$$

$$\mu_1 = (k_{20} - k_{10} - 2i\sqrt{k_{10}k_{20}})\,\varepsilon + O(\varepsilon^2), \qquad \mu_3 = \frac{1}{4}\left[k_{10}^2 + \frac{1}{2}k_{10}\left(\frac{dk_1}{d\varepsilon}\right)_0\right]\varepsilon + O(\varepsilon^2) \tag{5.2}$$
$$\mu_2 = (k_{20} - k_{10} + 2i\sqrt{k_{10}k_{20}})\,\varepsilon + O(\varepsilon^2),$$

The subscript zero indicates that the corresponding value was calculated at $\epsilon = 0$, i.e. for the original sphere. The quantity $(dk_1/d\epsilon)_0$ can be easily calculated

$$\frac{dk_1}{d\varepsilon} = \frac{1}{A_1^2}\left(A_1\frac{dC_1}{d\varepsilon} - C_1\frac{dA_1}{d\varepsilon}\right)$$

To evaluate $dA_1/d\epsilon$ and $dC_1/d\epsilon$ at $\epsilon = 0$, one can determine the changes of these moments relative to changes of the cavity shape, from spherical into ellipsoidal, and then evaluate the limits

$$\lim_{\varepsilon\to 0}\frac{\Delta A_1}{\varepsilon}=\left(\frac{dA_1}{d\varepsilon}\right)_0, \qquad \lim_{\varepsilon\to 0}\frac{\Delta C_1}{\varepsilon}=\left(\frac{dC_1}{d\varepsilon}\right)_0$$

The computations yield

$$\frac{dk_1}{d\varepsilon}=-\frac{1}{3}\frac{\gamma_1}{\gamma}k_{20}(2+k_{10}) \tag{5.3}$$

(γ is the density of the liquid, γ_1 is the density of the shell).

It follows from the obtained expressions (5.2) that two of the roots of inequality (4.5) are complex for small values of ϵ. The inequality evidently is fulfilled when $\mu > \mu_3$, i.e. the motion is unstable when

$$r_0^2<\frac{k_3}{\mu_3} \tag{5.4}$$

This indicates that in such a case the qualitative picture remains the same as for the rigid body. It possesses only one critical angular velocity. Nevertheless, it was not proved that the motion will be stable at $r_0^2 > k_3/\mu_3$, because we have only investigated the linearized equations here.

6. Although the qualitative picture appeared to be the same as for the rigid body in all the cases considered above, still, since the instability condition (4.5) is expressed mathematically as a cubic inequality (the corresponding inequality for a rigid body is linear), one can expect new cases: absolute instability (if inequality (4.5) does not have positive real roots) or the appearance of a second region of instability at the larger angular velocities (when two or three positive roots of inequality (4.5) are present.

For example, at $\epsilon = -0.5$, $k_1 = 0.5$, the inequality (4.5) becomes

$$\mu^3-0.0850\ \mu^2-0.0350\ \mu+0.061>0$$

and it is satisfied everywhere (one negative root and two complex). The motion of such a system is always unstable. Nevertheless, the problem concerning the construction of a real gyroscope containing the cavity with k_1, k_2 and ϵ prescribed, although being theoretically fully realizable (there are only three equations to deal with for 6 to 7 selected quantities: three parameters characterizing the cavity, density of the shell and the fluid and parameters defining the external contour of the body), is extremely difficult to cope with in practice, and therefore no examples of such "anomalous" gyroscopes are presented in this paper.

BIBLIOGRAPHY

1. Greenhill, A.G., On the rotation of a liquid ellipsoid about its mean axis. *Proc. Camb. Phil. Soc.* 3, 1879.

2. Hough, ., The oscillations of a rotating ellipsoidal shell containing fluid. *Phil. Trans. A.* 186, 1895.

3. Poincare, H., Sur la precession des corps deformables. *Bull. Astr.* 27, 1910.

4. Zhukovskii, N.E., O dvizhenii tverdogo tela, imeiushchego polosti, napolnennye odnorodnoi kapel'noi zhidkost'iu (On the motion of a rigid body containing a cavity filled with a homogeneous liquid). *Sobr. soch.* Vol. 2, 1949.

5. Chetaev, N.G., Ob odnoi zadache Koshi (On a problem of Cauchy). *PMM* Vol. 9, No. 2, 1945.

6. Chetaev, N.G., Ob ustoichivosti vrashcheniia tverdogo tela s odnoi nepodvizhnoi tochkoi v sluchaie Lagranzha (On the stability of rotation of a rigid body with one fixed point in the Lagrange case). *PMM* Vol. 18, No. 1, 1954.

7. Rumiantsev, V.V., Ob ustoichivosti vrashcheniia tiazhelogo tela s odnoi nepodvizhnoi tochkoi v sluchaie S.V. Kovalevskoi (On the stability of rotation of a heavy rigid body with one fixed point in the S.V. Kovalevskaia case). *PMM* Vol. 18, No. 4, 1954.

8. Liapunov, A.M., *Obshchaia zadacha ob ustoichivosti dvisheniia (The General Problem of the Stability of Motion)*. Moscow-Leningrad, 1935.

9. Zhak, S.V., O vozmozhnosti kvazitverdogo dvizheniia zhidkosti (About the possibility of quasi-rigid motion of a liquid). *PMM* Vol. 21, No. 4, 1957.

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